

# A SYSTEM OF SIX RECTANGULAR BIQUADRATICS

*Dissertation*

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS  
UNIVERSITY IN CONFORMITY WITH THE REQUIREMENTS FOR  
THE DEGREE OF DOCTOR OF PHILOSOPHY

1929

BY

MILDRED WATERS DEAN

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# A System of Six Rectangular Biquadratics.

BY MILDRED WATERS DEAN.

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This paper discusses a special set of six points in the inversive plane, and certain biquadratic curves related to them. It takes up the following topics:

I. Representing the three vertices of a triangle by complex quantities  $a$ ,  $b$ , and  $c$ , and studying the locus traced out by a point moving so as to subtend equal angles (modulo  $\pi$ ) at two sides  $ab$  and  $ac$  of the triangle, we know that this locus is a biquadratic having a double point at  $a$ , cutting itself at right angles there, and passing through the other two vertices of the triangle, its two Fermat points (denoted by  $f$  and  $f_1$ ) and the point  $\infty$ . Then, in addition to the biquadratic cutting itself at right angles at  $a$ , and passing through  $b$ ,  $c$ ,  $f$ ,  $f_1$  and  $\infty$ , there are five other curves which bear the same relation to the set of six points  $a$ ,  $b$ ,  $c$ ,  $f$ ,  $f_1$  and  $\infty$ , namely: five biquadratics cutting themselves at right angles at  $b$ ,  $c$ ,  $f$ ,  $f_1$  and  $\infty$ , respectively, and passing through the other five points of the set. Thus, the three vertices of the triangle, the Fermat points, and the point  $\infty$  are a special set of six points such that there can be found six biquadratic curves, each having a double point at one of the six points, cutting itself at right angles there, and passing through the other five.

Moreover, it is shown that this set of six points can be broken up into two triads which are apolar.

II. The study of the conditions under which, given a set of six points, the biquadratic having a double point at each and passing through the others will be rectangular. (A rectangular curve is one cutting itself at right angles at a double point.) A set of six points in the inversive plane is chosen, with the restriction that one of them be the point  $\infty$ , and the necessary conditions are derived.

III. We next give the geometrical interpretation of the conditions derived in Section II. It is known that five points have a biquadratic invariant  $I_2$ .\* For any given four points this is a covariant biquadratic curve  $C_2$ . The relation of the four and any point on  $C_2$  is symmetrical. It is found that the set of six points mentioned in Section II is such that  $I_2$  vanishes for any five, or, in other words, that the covariant  $C_2$  of any four is on the other two.

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\* This invariant  $I_2$  is given in a "Note on Neuberg's Cubic Curve," by Professor Morley, *American Mathematical Monthly*, Vol. 32 (1925), pp. 407-411.



IV. It is then pointed out that the six points, together with the associated biquadratics, may be taken as the fundamental points and curves of a Geiser involution in the inversive plane.

## I.

We have before us the problem of finding the locus of a point which moves so as to subtend equal angles, modulo  $\pi$ , at two sides of a given triangle  $abc$ .\* (By "equal modulo  $\pi$ " we mean that either the angles are equal, or they differ by  $\pi$ .) If we denote our moving point by a complex quantity  $z$ , we may first try to find its locus when it moves so as to subtend equal angles (modulo  $\pi$ ) at the sides  $ab$  and  $ac$ . The line from  $b$  to  $z$  is a vector  $z - b = \rho_1 e^{i\theta_1}$ . Likewise, the line from  $a$  to  $z$  is a vector  $z - a = \rho_2 e^{i\theta_2}$  and that from  $c$  to  $z$  is a vector  $z - c = \rho_3 e^{i\theta_3}$ . Now we wish the angles  $azb$  and  $cza$  to be equal, or else to differ by  $\pi$ . If we take the quotient  $(z - b)/(z - a)$ , we find that it is equal to  $(\rho_1/\rho_2)e^{i(\theta_1 - \theta_2)}$ , where  $\theta_1 - \theta_2$  is the angle  $azb$ . Similarly,  $(z - a)/(z - c)$  is equal to  $(\rho_2/\rho_3)e^{i(\theta_2 - \theta_3)}$ , where  $\theta_2 - \theta_3$  is the angle  $cza$ . Now form the quotient

$$[(z - b)/(z - a)]/[(z - a)/(z - c)] = (\rho_1\rho_3/\rho_2^2)e^{i(\theta_1 + \theta_3 - 2\theta_2)}.$$

If the angles  $azb$  and  $cza$  are to be equal, we must have  $\theta_1 - \theta_2 = \theta_2 - \theta_3$ , or  $\theta_1 + \theta_3 - 2\theta_2 = 0$ . If  $azb - cza = \pi$ , we have  $\theta_1 - 2\theta_2 + \theta_3 = \pi$ . Thus, the exponent of "e" in the quotient  $[(z - b)/(z - a)]/[(z - a)/(z - c)]$  either vanishes or reduces to  $i\pi$ , so that, in either case, the quotient reduces to a real quantity. Therefore, we have as a first fundamental equation:

$$(1) \quad [(z - b)/(z - a)]/[(z - a)/(z - c)] = \text{a real quantity.}$$

But, since a real quantity always equals its conjugate, we may write

$$(2) \quad [(z - b)/(z - a)]/[(z - a)/(z - c)] \\ = [(\bar{z} - \bar{b})/(\bar{z} - \bar{a})]/[(\bar{z} - \bar{a})/(\bar{z} - \bar{c})],$$

$$(3) \quad (z - a)^2(\bar{z} - \bar{b})(\bar{z} - \bar{c}) - (\bar{z} - \bar{a})^2(z - b)(z - c) = 0.$$

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\* This is, of course, an old problem. See, for example, Salmon's "Higher Plane Curves." The problem is also discussed in an article by Professor M. T. Naranjangar, in the *Proceedings of the Edinburgh Mathematical Society*, Vol. 28, p. 73. The curve was called by the Belgian geometers the "focale à noeud." It is identical with the strophoid, which was first considered by Barrow in *Lectiones Geometricae* (1669), p. 69. Further references are: Van Rees, *Correspondance Mathématique de Quetelet*, tome V, p. 361; Lebon, *Journal de Mathématiques Spéciales* (1895); Teixeira, *Traité des Courbes Spéciales Remarquables*, tome I, p. 30. A history of the problem is given in Loria's *Speciale Algebraische und Transcendente Ebene Kurven* (1902), pp. 66-67, where various additional references are given.

The curve given by (3) is, inversively speaking, a biquadratic passing through  $\infty$ ; projectively speaking, it is a circular cubic.\* We shall think of it as a biquadratic. It is obvious that the curve passes through  $a$ ,  $b$ , and  $c$ . Since the two Fermat points subtend equal angles, modulo  $\pi$ , at the sides of the given triangle, it is evident that our curve also passes through these points.

If we consider the point  $z$  in a position very near to  $a$ , we know, of course, by hypothesis, that angle  $azb = \text{angle } cza$ . The angles  $abz$  and  $acz$  have become infinitesimal, and may be considered as equal. Then the angle  $baz = \text{angle } zac$ ; that is, the angles at the vertex  $a$  become more nearly equal as  $z$  approaches  $a$ . This shows that the curve (3), passing through  $a$ , bisects the angle at  $a$ . In fact, it can be shown that this curve not only bisects the angle at  $a$ , but it has a double point there, and cuts itself at right angles at this point. Let us denote this curve by  $A$ .

But  $A$  is not the only curve which furnishes us with a solution of our problem. For we may also have

$$(4) \quad (z - c)^2(\bar{z} - \bar{a})(\bar{z} - \bar{b}) - (\bar{z} - \bar{c})^2(z - a)(z - b) = 0.$$

This is the locus of a point  $z$  which moves so as to subtend equal angles, mod.  $\pi$ , at the sides  $ac$  and  $bc$ . It is a biquadratic on  $\infty$  which passes through  $c$ , cuts itself at right angles there, and bisects the angle at  $c$ . It also passes through  $a$  and  $b$  and the two Fermat points. We denote this curve by  $C$ . Similarly,

$$(5) \quad (z - b)^2(\bar{z} - \bar{a})(\bar{z} - \bar{c}) - (\bar{z} - \bar{b})^2(z - a)(z - c) = 0$$

will be the locus of a point  $z$  moving so as to subtend equal angles, mod.  $\pi$ , at the sides  $bc$  and  $ba$ . It is also a biquadratic passing through  $a$ ,  $b$ ,  $c$ ,  $f$ ,  $f_1$  and  $\infty$ . It cuts itself at right angles at  $b$ , and bisects the angle at  $b$ . We shall speak of this curve as the curve  $B$ .

So we have found three different curves which are solutions of our problem. We thus have a net of biquadratics on the six points  $a$ ,  $b$ ,  $c$ ,  $f$ ,  $f_1$  and  $\infty$ . Now, since our biquadratics have double points at  $a$ ,  $b$ , and  $c$ , respectively, and cut themselves at right angles at these points, is it not natural to suppose that we can find three more curves of the net, which have double points at  $f$ ,  $f_1$  and  $\infty$ , respectively, and which cut themselves at right angles at these points? This assumption is found to be true.

No simple method of obtaining the equations of the two biquadratics having double points at  $f$  and  $f_1$  has been found, but we may find the equation of the one having a double point at  $\infty$  as follows: the curve will have an

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\* See explanatory note at the end of this paper.



equation of the form  $LA + MB + NC = 0$ , where  $L$ ,  $M$ , and  $N$  are real constants, which are to be determined. They must be determined in such a way that the curve will cut itself at right angles at  $\infty$ . This means that the coefficients of the terms in  $z^2\bar{z}^2$ ,  $z^2\bar{z}$ ,  $z\bar{z}^2$  and  $z\bar{z}$  must all be zero. The coefficient of  $z^2\bar{z}^2$  is zero already. Equating the coefficient of  $z^2\bar{z}$  to zero we have

$$(6) \quad L(2\bar{a} - \bar{b} - \bar{c}) + M(2\bar{b} - \bar{a} - \bar{c}) + N(2\bar{c} - \bar{a} - \bar{b}) = 0.$$

Equating the coefficient of  $z\bar{z}^2$  to zero we have

$$(7) \quad L(b + c - 2a) + M(a + c - 2b) + N(a + b - 2c) = 0.$$

Equating the coefficient of  $z\bar{z}$  to zero,

$$(8) \quad L[2a(\bar{b} + \bar{c}) - 2\bar{a}(b + c)] + M[2b(\bar{a} + \bar{c}) - 2\bar{b}(a + c)] \\ + N[2c(\bar{a} + \bar{b}) - 2\bar{c}(a + b)] = 0.$$

In (6), (7), and (8) we have three linear homogeneous equations in three unknowns  $L$ ,  $M$ , and  $N$ . They have solutions different from zero only if

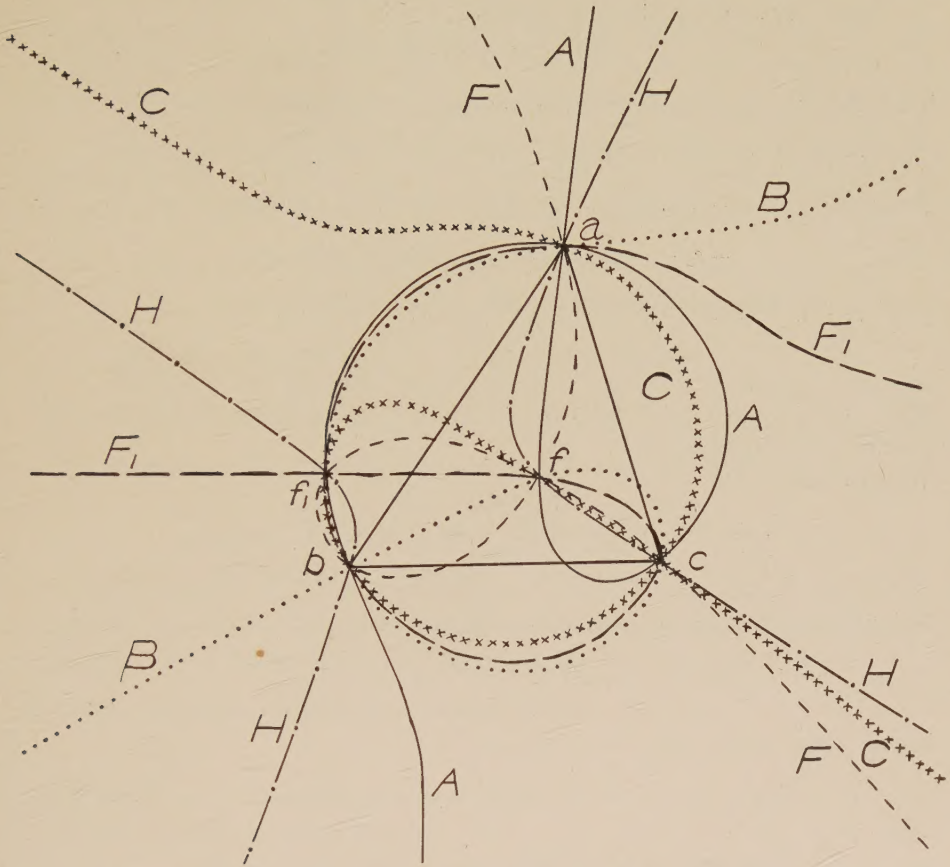
$$\begin{vmatrix} 2\bar{a} - \bar{b} - \bar{c} & 2\bar{b} - \bar{a} - \bar{c} & 2\bar{c} - \bar{a} - \bar{b} \\ b + c - 2a & a + c - 2b & a + b - 2c \\ 2a(\bar{b} + \bar{c}) - 2\bar{a}(b + c) & 2b(\bar{a} + \bar{c}) - 2\bar{b}(a + c) & 2c(\bar{a} + \bar{b}) - 2\bar{c}(a + b) \end{vmatrix} = 0$$

Adding the sum of the second and third columns to the first makes each element of the first column zero. Obviously, then, the values  $L = M = N = 1$  will satisfy equations (6), (7), and (8), so that our biquadratic having a double point at infinity and cutting itself at right angles there is  $A + B + C = 0$ . A biquadratic cutting itself at right angles at infinity is known to be a rectangular hyperbola. In our case, the curve also passes through the three vertices of the triangle and its two Fermat points. The rectangular hyperbola passing through the three vertices of a triangle and its two Fermat points is known as the Kiepert hyperbola of the triangle. So our curve is nothing more than the Kiepert hyperbola of the triangle.

We may now state the following theorem:

**THEOREM 1.** *The three vertices of a triangle, its two Fermat points, and the point infinity form a set of six points such that the biquadratic having a double point at any one of them, and passing through the others, is rectangular.*





THE SIX RECTANGULAR BIQUADRATICS ASSOCIATED WITH THE TRIANGLE  $abc$ .

|                                                     |                   |
|-----------------------------------------------------|-------------------|
| CURVE $A$ —Biquadratic with Double Point at $a$     | —————             |
| CURVE $B$ —Biquadratic with Double Point at $b$     | .....             |
| CURVE $C$ —Biquadratic with Double Point at $c$     | x x x x x x x x x |
| CURVE $F$ —Biquadratic with Double Point at $f$     | —————             |
| CURVE $F_1$ —Biquadratic with Double Point at $f_1$ | —————             |
| CURVE $H$ —Kiepert Hyperbola                        | ——— . ——— . ———   |

In addition, these points fall into two triads  $abc$  and  $ff_1 \infty$  which are apolar. If  $abc$  and  $ff_1 \infty$  are to be apolar, we must have

$$3abc - (f + f_1 + \infty)(ab + ac + bc) + (a + b + c)(ff_1 + f\infty + f_1\infty) - 3ff_1\infty = 0.$$

Then the sum of all quantities multiplying  $\infty$  must equal zero, or

$$-(ab + ac + bc) + (a + b + c)f + f_1 - 3ff_1 \text{ should equal zero.}$$

Replacing  $a + b + c$  by  $s_1$  and  $ab + ac + bc$  by  $s_2$  we write:

$$(9) \quad -s_2 + s_1(f + f_1) - 3ff_1 \text{ should equal zero.}$$

$f$  and  $f_1$  may be expressed in terms of  $a, b$ , and  $c$  as follows:

$$f = s_1/3 - \bar{v}_1 v_2 / 3\bar{v}_2, \quad f_1 = s_1/3 - v_1 \bar{v}_2 / 3\bar{v}_1,$$

where  $v_1 = a + \omega b + \omega^2 c$  and  $v_2 = a + \omega^2 b + \omega c$  ( $\omega$  being one of the complex cube roots of unity.)

We have now

$$\begin{aligned} f + f_1 &= 2s_1/3 - (\bar{v}_1^2 v_2 + \bar{v}_2^2 v_1) / 3\bar{v}_1 v_2, \\ ff_1 &= s_1^2/9 + v_1 v_2 / 9 - s_1/9 (\bar{v}_1^2 v_2 + \bar{v}_2^2 v_1 / \bar{v}_1 \bar{v}_2). \end{aligned}$$

Substituting these values in equation (9) we have

$$\begin{aligned} -s_2 + 2s_1^2/3 - (s_1/3) [(\bar{v}_1^2 v_2 + \bar{v}_2^2 v_1) / \bar{v}_1 \bar{v}_2] \\ - s_1^2/3 - v_1 v_2 / 3 + (s_1/3) [(\bar{v}_1^2 v_2 + \bar{v}_2^2 v_1) / \bar{v}_1 \bar{v}_2] \end{aligned}$$

should equal zero.

$$s_1^2/3 - s_2 - v_1 v_2 / 3 \text{ should equal zero.}$$

Now

$$v_1 v_2 = (a + \omega^2 b + \omega c)(a + \omega b + \omega^2 c) = s_1^2 - 3s_2.$$

Then we have

$$s_1^2/3 - s_2 - s_1^2/3 + s_2 = 0;$$

which shows that the triads  $abc$  and  $ff_1 \infty$  are apolar. This gives us

**THEOREM 2.** *The three vertices of a triangle are a set of points which is apolar to the set consisting of the two Fermat points of the triangle and the point infinity.*

## II.

We now take, on a curve  $K$  (namely, a biquadratic which cuts itself at right angles), the double point and five other points, and derive the condition

under which the biquadratic having a double point at each and passing through the others, will be rectangular. If we choose  $K$  to be a biquadratic cutting itself at right angles at infinity, we know that this is a rectangular hyperbola, and since the other five points must lie on this curve, we see that we may take as their co-ordinates  $x_i = m_i$   $y_i = 1/m_i$  ( $i = 1, 2, 3, 4, 5$ ). Here  $x$  and  $y$  are Cartesian co-ordinates, and are not complex.

If we wish to write the biquadratic passing through one of the points, say  $x_1 y_1$ , which also passes through  $\infty$ , we need only write the equation of a circular cubic passing through  $x_1 y_1$ . This will be of the form

$$(10) \quad [(x - x_1)^2 + (y - y_1)^2] [a(x - x_1) + b(y - y_1)] \\ + c(x - x_1)^2 + d(x - x_1)(y - y_1) + e(y - y_1)^2 \\ + f(x - x_1) + g(y - y_1) = 0,$$

( $a, b$ , and  $c$  are real, arbitrary constants, and not the  $a, b$ , and  $c$  of the previous section.) When  $x_1 = m_1$  and  $y_1 = 1/m_1$  this cuts the hyperbola when

$$(11) \quad [(m - m_1)^2 + (1/m - 1/m_1)^2] [a(m - m_1) + b(1/m - 1/m_1)] \\ + c(m - m_1)^2 + d(m - m_1)(1/m - 1/m_1) \\ + e(1/m - 1/m_1)^2 + f(m - m_1) + g(1/m - 1/m_1) = 0.$$

Since this is to have a double point at  $m_1, 1/m_1$  and cut itself at right angles there, we must have  $f = g = 0$  and  $e = -c$ . Putting these values in equation (11), simplifying, and removing the factor  $(m - m_1)^2$ , we have

$$(12) \quad [m^2 m_1^2 + 1] [amm_1(m - m_1) - b(m - m_1)] \\ + cm^3 m_1^3 - dm^2 m_1^2 - cmm_1 = 0.$$

This is obviously a quartic in  $m$ , the coefficient of  $m^4$  being  $am_1^3$ . We may accordingly write equation (12) as

$$[m^2 m_1^2 + 1] [amm_1(m - m_1) - b(m - m_1)] + cm^3 m_1^3 \\ - dm^2 m_1^2 - cmm_1 \equiv am_1^3 (m - a_1)(m - a_2)(m - a_3)(m - a_4),$$

pass through the points  $m_2, 1/m_2; m_3, 1/m_3; m_4, 1/m_4$  and  $m_5, 1/m_5$ , where  $a_1, a_2, a_3$  and  $a_4$  are the roots of the quartic. But we wish our curve to Accordingly, we may substitute  $m_2, m_3, m_4$  and  $m_5$  for  $a_1, a_2, a_3$ , and  $a_4$  in the last equation, which then becomes

$$(13) \quad [m^2 m_1^2 + 1] [amm_1(m - m_1) - b(m - m_1)] + cm^3 m_1^3 \\ - dm^2 m_1^2 - cmm_1 \equiv am_1^3 (m - m_2)(m - m_3)(m - m_4)(m - m_5).$$

We are trying to find the conditions under which the biquadratic having a double point at  $m_1, 1/m_1$  and passing through the four other points  $m_2, 1/m_2; m_3, 1/m_3; m_4, 1/m_4; m_5, 1/m_5$  and the point at infinity will be



rectangular. This will amount to finding some relation on  $m_1, m_2, m_3, m_4$  and  $m_5$ . In (13) we have a relation involving all five  $m$ 's, but it also involves the arbitrary constants  $a, b, c$  and  $d$ . We must get rid of these constants in some way. To do this, we take equation (13) and let  $m = m_1$ ,  $mm_1 = i$  and  $mm_1 = -i$  successively, thus obtaining the three equations

$$(14) \quad c(m_1^6 - m_1^2) - dm_1^4 = am_1^3(m_1^4 - m_1^3\sigma_1 + m_1^2\sigma_2 - m_1\sigma_3 + \sigma_4),$$

$$(15) \quad -2ci + d = (a/m_1)(i^4 - i^3m_1\sigma_1 + i^2m_1^2\sigma_2 - im_1^3\sigma_3 + m_1^4\sigma_4),$$

$$(16) \quad 2ci + d = (a/m_1)(i^4 + i^3m_1\sigma_1 + i^2m_1^2\sigma_2 + im_1^3\sigma_3 + m_1^4\sigma_4),$$

where  $\sigma_1, \sigma_2, \sigma_3$  and  $\sigma_4$  are the four symmetric functions of  $m_2, m_3, m_4$  and  $m_5$ .

Solving the simultaneous equations (15) and (16) we obtain  $c = a/2(m_1^2\sigma_3 - \sigma_1)$  and  $d = a/m_1(1 - m_1^2\sigma_2 + m_1^4\sigma_4)$ . Substituting these back in equation (14), we find that each term contains  $a$  as a factor, so that this may be cancelled out, leaving an expression which involves only  $m_1$  and the symmetric functions of the other four  $m$ 's. This expression is, after some reductions have been made,

$$(17) \quad \sigma_1 + m_1^2\sigma_3 - 2m_1\sigma_4 - 2m_1 = 0.$$

This equation involves  $m_1$  in a way different from that in which the remaining four  $m$ 's are involved. This is because we started out by letting our biquadratic have a double point at  $m_1, 1/m_1$ . But we wish to treat all the  $m$ 's alike; i. e., we wish to find the condition under which the biquadratic having a double point at *any*  $m$  and passing through the others will be rectangular. So we must find a relation which will treat all the  $m$ 's alike. To do this, we introduce the symmetric functions of  $m_1, m_2, m_3, m_4$  and  $m_5$ , which are denoted by  $s_1, s_2, s_3, s_4$  and  $s_5$ , and which are related to the  $\sigma$ 's as follows:

$$\begin{aligned} s_1 &= m_1 + \sigma_1, & s_3 &= m_1\sigma_2 + \sigma_3, & s_5 &= m_1\sigma_4, \\ s_2 &= m_1\sigma_1 + \sigma_2, & s_4 &= m_1\sigma_3 + \sigma_4, \end{aligned}$$

Also, we have

$$\sigma_1 = s_1 - m_1, \quad \sigma_3 = (s_4m_1 - s_5)/m_1^2, \quad \sigma_4 = s_5/m_1.$$

Putting these into equation (17) and simplifying, we have:

$$(18) \quad m_1(s_4 - 3) + s_1 - 3s_5 = 0.$$

Equating coefficients of like powers of  $m_1$  gives

$$(19) \quad s_4 = 3, \quad s_1 = 3s_5.$$

Equations (19) give the condition which we sought; it proves to be two

relations on  $m_1, m_2, m_3, m_4$  and  $m_5$ ; if we desire to have just one condition, we may effect this by substituting for the  $s$ 's their values in terms of the  $\sigma$ 's; this gives:  $m_1 \sigma_3 + \sigma_4 = 3$  and  $m_1 + \sigma_1 = 3m_1 \sigma_4$ . Elimination of  $m_1$  from these two equations gives

$$(20) \quad \sigma_1 \sigma_3 - 10\sigma_4 + 3\sigma_4^2 + 3 = 0.$$

This is a relation on  $m_2, m_3, m_4$  and  $m_5$ , but it will hold for any four of the  $m$ 's. For instance, we might have started working with a biquadratic having a double point at  $m_2$ , and then we should have arrived at a relation (20) involving  $m_1, m_3, m_4$  and  $m_5$ . So (20) holds for any four of the  $m$ 's.

Equation (20) is fundamental in this work, so we state

**THEOREM 3.** *If we take, on a biquadratic which cuts itself at right angles at infinity, the point infinity and five other points whose co-ordinates are  $m_1, 1/m_1; m_2, 1/m_2; m_3, 1/m_3; m_4, 1/m_4$  and  $m_5, 1/m_5$ , then the condition that the biquadratic having a double point at  $m_1, 1/m_1$  and passing through the other five points shall be rectangular is:  $\sigma_1 \sigma_3 - 10\sigma_4 + 3\sigma_4^2 + 3 = 0$ , where the  $\sigma$ 's are the symmetric functions of  $m_2, m_3, m_4$  and  $m_5$ .*

### III.

In order to get a geometrical interpretation of the condition given in equation (20), we investigate the connection between our six points and Neuberg's cubic curve for three points.

This curve is defined as follows: starting out with three points  $x_1 y_1, x_2 y_2, x_3 y_3$ , Neuberg's cubic is the locus of points  $x_4 y_4$  satisfying the condition.\*

$$\Delta' \equiv \begin{vmatrix} \lambda_{23}\lambda_{14} & \lambda_{23} + \lambda_{14} & 1 \\ \lambda_{31}\lambda_{24} & \lambda_{31} + \lambda_{24} & 1 \\ \lambda_{12}\lambda_{34} & \lambda_{12} + \lambda_{34} & 1 \end{vmatrix} = 0,$$

where  $\lambda_{23}$  is the squared distance between the points  $x_2 y_2$  and  $x_3 y_3$ , and similarly for the other  $\lambda$ 's. This curve passes through the points  $x_1 y_1, x_2 y_2$  and  $x_3 y_3$ . In other words, if four points lie on a Neuberg cubic,  $\Delta'$  vanishes for them.

If we subject the four points involved in  $\Delta'$  to an inversion with center at  $x_5 y_5$  and radius equal to  $k$ , we get

$$\Delta' = k^{12} I_2 / \lambda_{15}^2 \lambda_{25}^2 \lambda_{35}^2 \lambda_{45}^2,$$

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\* See the note by Professor Morley, referred to above.

where

$$I_2 \equiv \begin{vmatrix} \lambda_{23}\lambda_{14} & \lambda_{23}\lambda_{15}\lambda_{45} + \lambda_{14}\lambda_{25}\lambda_{35} & 1 \\ \lambda_{31}\lambda_{24} & \lambda_{31}\lambda_{25}\lambda_{45} + \lambda_{24}\lambda_{35}\lambda_{15} & 1 \\ \lambda_{12}\lambda_{34} & \lambda_{12}\lambda_{35}\lambda_{45} + \lambda_{34}\lambda_{15}\lambda_{25} & 1 \end{vmatrix} = 0.$$

If  $\Delta' = 0$  is the original Neuberg cubic, then  $I_2 = 0$  is a new curve, which is the locus of a point  $x_4y_4$ , the other four points being given. The curve whose equation is  $I_2 = 0$  is a biquadratic; it is a covariant biquadratic curve of the four points  $x_1, y_1; x_2, y_2; x_3, y_3$  and  $x_5, y_5$ . This curve, which may be denoted by  $C_2$ , is the "inverted Neuberg cubic with respect to one of the points (say  $x_5, y_5$ ) of any three of the remaining four (say  $x_1, y_1; x_2, y_2$  and  $x_3, y_3$ ). The fifth point  $x_4, y_4$  is the variable point which traces out the locus."\* If five points lie on an inverted Neuberg cubic,  $I_2$  must vanish for them.

It should be noticed that while the Neuberg cubic seems to involve only four points, it really involves five, since it is a biquadratic on infinity. So it involves the four points which are explicitly contained in it, plus the point infinity. The inverted Neuberg cubic, or better, covariant biquadratic  $C_2$  involves five points explicitly, since, on inversion, infinity goes into  $x_5, y_5$ , and this appears in the equation.

After this digression on the Neuberg cubic and the covariant biquadratic  $C_2$ , we discuss the connection between these curves and our set of six points.

If we wish four of our points, say  $m_2, 1/m_2; m_3, 1/m_3; m_4, 1/m_4$ , and  $m_5, 1/m_5$  to lie on a Neuberg cubic, we must have

$$\begin{vmatrix} \left[ (m_3 - m_4)^2 + \frac{(m_4 - m_3)^2}{m_4^2 m_3^2} \right] \left[ (m_2 - m_5)^2 + \frac{(m_5 - m_2)^2}{m_5^2 m_2^2} \right] & (m_3 - m_4)^2 + \frac{(m_4 - m_3)^2}{m_4^2 m_3^2} + (m_2 - m_5)^2 + \frac{(m_5 - m_2)^2}{m_5^2 m_2^2} & 1 \\ \left[ (m_4 - m_2)^2 + \frac{(m_2 - m_4)^2}{m_2^2 m_4^2} \right] \left[ (m_3 - m_5)^2 + \frac{(m_5 - m_3)^2}{m_3^2 m_5^2} \right] & (m_4 - m_2)^2 + \frac{(m_2 - m_4)^2}{m_2^2 m_4^2} + (m_3 - m_5)^2 + \frac{(m_5 - m_3)^2}{m_3^2 m_5^2} & 1 \\ \left[ (m_2 - m_3)^2 + \frac{(m_3 - m_2)^2}{m_2^2 m_3^2} \right] \left[ (m_4 - m_5)^2 + \frac{(m_5 - m_4)^2}{m_4^2 m_5^2} \right] & (m_2 - m_3)^2 + \frac{(m_3 - m_2)^2}{m_2^2 m_3^2} + (m_4 - m_5)^2 + \frac{(m_5 - m_4)^2}{m_4^2 m_5^2} & 1 \end{vmatrix} = 0$$

The problem of evaluating this determinant is quite interesting in itself, but since it involves many detailed operations which would be tedious for the

\* Morley, *loc. cit.*, page 408.



reader, and which are unimportant for the purpose of this paper, we may omit them, and let it suffice to say that the determinant may finally be reduced to

$$(m_2 m_3 m_4 m_5 - 1)^2 \begin{vmatrix} (A-B)^2 & C & 1 \\ (A-C)^2 & B & 1 \\ (B-C)^2 & A & 1 \end{vmatrix} + \begin{vmatrix} C^2(A-B)^2 & C & 1 \\ B^2(A-C)^2 & B & 1 \\ A^2(B-C)^2 & A & 1 \end{vmatrix} = 0,$$

where

$$A = m_2 m_3 + m_4 m_5, \quad B = m_2 m_4 + m_3 m_5, \quad C = m_2 m_5 + m_3 m_4.$$

This gives

$$(\sigma_4 - 1)^2 [-3(A-B)(B-C)(C-A)] \\ - (AB + AC + BC)(A-B)(B-C)(C-A) = 0.$$

Then either

$$3(\sigma_4 - 1)^2 + (AB + AC + BC) = 0 \text{ or } (A-B)(B-C)(C-A) = 0.$$

Making the second expression equal zero makes one of its factors zero. If, say,  $A - B = 0$ , then  $A = B$  and  $m_2 m_3 + m_4 m_5 = m_2 m_4 + m_3 m_5$ . This makes either  $m_5 = m_2$  or  $m_4 = m_3$ . We assumed our six points to be distinct, so this relation cannot hold. Therefore, we have

$$3(\sigma_4 - 1)^2 + (AB + AC + BC) = 0, \quad 3(\sigma_4 - 1)^2 + \sigma_1 \sigma_3 - 4\sigma_4 = 0, \\ 3\sigma_4^2 - 6\sigma_4 - 4\sigma_4 + \sigma_1 \sigma_3 + 3 = 0, \quad \sigma_1 \sigma_3 - 10\sigma_4 + 3\sigma_4^2 + 3 = 0.$$

Thus the condition that four of the points lie on a Neuberg cubic is exactly the same as the condition that the biquadratic having a double point at one of the six points and passing through the others shall be rectangular, since in each case the condition is

$$\sigma_1 \sigma_3 - 10\sigma_4 + 3\sigma_4^2 + 3 = 0.$$

Since this holds for *any four* of our five points, we know that *any four* of them lie on a Neuberg cubic, or  $\Delta'$  vanishes for any four of our five finite points. However, this really involves not four but *five* points. For when we are given a set of  $n$  points in the inversive plane, we have, in reality,  $n + 1$  points, for the point infinity may always be regarded as an additional point of the set. So this condition involves, in reality, five points of our set, namely, four of the five finite points and infinity. If we do not like to bother with the point infinity, we can invert with respect to a point  $a_6$ . So that now, instead of dealing with a set of points  $m_1, 1/m_1; m_2, 1/m_2; m_3, 1/m_3; m_4, 1/m_4; m_5, 1/m_5$  and infinity, we are dealing with a set of points  $a_1, a_2, a_3, a_4, a_5$  and  $a_6$ , where  $a_i (i = 1 \cdots 5)$  is the inverse of the point  $m_i, 1/m_i$

and  $a_6$  is the inverse of infinity. The old condition for rectangularity will invert into a new condition, and this will be the condition that the biquadratic having a double point at one of the six (finite) points  $a_1, \dots, a_6$  and passing through the others shall be rectangular. The old condition involved four points, and said that any four of the five finite points in the old set lay on a Neuberg cubic; the new condition involves five points and says that any *five* of the points  $a_1, \dots, a_6$  lie on a covariant curve  $C_2$  (since a Neuberg cubic inverts into a  $C_2$ ), or better, that  $I_2$  vanishes for any five of the six points.

Now, suppose we investigate more closely our set of points  $a_1, \dots, a_6$ . Four of these, say  $a_2, a_3, a_4$  and  $a_5$  will determine a covariant curve  $C_2$  which passes through  $a_2, a_3, a_4$  and  $a_5$ . *Any point on this curve*, together with the points  $a_2, a_3, a_4$  and  $a_5$ , will satisfy the condition  $I_2 = 0$ . We have already seen that any five of our set of points  $a_1, \dots, a_6$  satisfy the condition  $I_2 = 0$ . That is,  $I_2$  will be zero for the points  $a_1, a_2, a_3, a_4$  and  $a_5$  and also for the set of points  $a_6, a_2, a_3, a_4$  and  $a_5$ . From this it is evident that  $a_1$  and  $a_6$  lie on the curve determined by  $a_2, a_3, a_4$  and  $a_5$ ; that is, the  $C_2$  determined by  $a_2, a_3, a_4$  and  $a_5$  passes through  $a_1$  and  $a_6$ . In the same way we see that the  $C_2$  determined by *any four* of the points is on the other two.

We may also put it as follows:

Choose a set of five of the points, say  $a_2, a_3, a_4, a_5$  and  $a_6$ . Then

$a_2, a_3, a_4, a_5$  determine a  $C_2$  which passes through  $a_1$  and  $a_6$ ,  
 $a_2, a_3, a_4, a_6$  determine a  $C_2$  which passes through  $a_1$  and  $a_5$ ,  
 $a_2, a_3, a_5, a_6$  determine a  $C_2$  which passes through  $a_1$  and  $a_4$ ,  
 $a_2, a_4, a_5, a_6$  determine a  $C_2$  which passes through  $a_1$  and  $a_3$ ,  
 $a_3, a_4, a_5, a_6$  determine a  $C_2$  which passes through  $a_1$  and  $a_2$ .

It is to be noticed that all these  $C_2$ 's pass through  $a_1$ . Thus we may say: If we choose a set of five of the six points, then, by choosing sets of four points from this set of five points, we determine five  $C_2$ 's. All five of these  $C_2$ 's then pass through the sixth point.

The results of the last three paragraphs are the most important ones arrived at in this paper. Accordingly, we re-state them, for better emphasis, as

**THEOREM 4.** *If we have a set of six points, which are such that the biquadratic having a double point at each, and passing through the others is rectangular, then the invariant  $I_2$  vanishes for any five of them. The covariant curve  $C_2$  determined by any four is on the other two. Moreover, if we choose a set of five of the six points, and set up the five covariant curves  $C_2$  which are determined by sets of four points chosen from this set of five points, then all these five  $C_2$ 's pass through the sixth point.*

## IV.

It is proper to regard the six points and associated biquadratics, which we have studied so far, as the fundamental points and curves of a Geiser involution in the inversive plane.

In the inversive plane, circles play the rôle which lines play in the projective plane. A Cremona transformation in this plane is one which transforms circles into curves which behave like circles; i. e., into curves  $C_n^*$  which are rational, which form a triply infinite system, and which are such that any two of them have as many *free* points of intersection as do two circles.

Professor Morley, starting from this definition,† has determined that there are four symmetric Cremona transformations in the inversive plane, and has given some of their properties. The transformations are:

- A. One which transforms circles into circles,
- B. One which transforms circles into  $C_3$ 's with four double points,
- C. One which transforms circles into  $C_7$ 's with six four-fold points,
- D. One which transforms circles into  $C_{15}$ 's with seven eight-fold points.

These transformations are all involutory.

Type C involves six points, and is therefore the one to which we shall confine our attention. This transformation sends a circle into a  $C_7$  with six four-fold points. These six points are the fundamental points of the transformation. What are the fundamental curves? They are, of course, the transforms of the fundamental points. The transformation sends the  $C_7$  on which a given fundamental point  $F$  is situated into a circle. Since the point  $F$  is of multiplicity four, a point  $P$ , in describing  $C_7$ , passes through  $F$  four times. The transform of  $P$ , called  $P'$ , describes a circle, and this circle must meet the transform of  $F$ , which is a curve  $C_n$ , just four times. A circle and a  $C_n$  meet  $2n$  times; therefore,  $2n = 4$ ,  $n = 2$ , and the fundamental curve is a biquadratic, or  $C_2$ . As a matter of fact, it can be shown that the biquadratic corresponding to a given fundamental point has a double point at that point, and passes through the remaining five fundamental points. Thus our fundamental system consists of six points and six biquadratics, each biquadratic having a double point at one of the six points, and passing through the others.

To a general circle corresponds then a  $C_7$  with four-fold points  $a_i$

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\* See explanatory note at end of paper.

† Unpublished report in mathematics seminary, Johns Hopkins University, December, 1926. The question as to whether a transformation so defined is one-to-one is not of interest here; it was, however, discussed by Professor Morley in the report mentioned here.



( $i=1, \dots, 6$ ). When the circle is on  $a_1$ , the corresponding fundamental biquadratic  $A_1$  is a factor of  $C_7$ ; the circle then corresponds to the other factor, a  $C_5$  with a double point at  $a_1$  and triple points at the other  $a$ 's.

When the circle is also on  $a_2$ , then  $A_2$  factors out from  $C_5$ , and what corresponds to the circle is a  $C_3$ , with  $a_1, a_2$  as simple points and  $a_3 a_4 a_5 a_6$  double points.

Finally, to the circle on  $a_1, a_2, a_3$  corresponds the circle on  $a_4, a_5, a_6$ .

To what Cremona transformation in the projective plane does this transformation correspond? In the projective plane, a Cremona transformation transforms lines into curves which behave like lines. So we must see what the effect of our transformation on a line will be. In the inversive plane, a line is simply a circle on infinity. Let us put one of the fundamental points of our transformation at infinity. Then, by (1) above, the transform of a line will be a  $C_5$  with infinity as a double point, and five triple points. But a  $C_5$  with infinity as a double point is, projectively, an octavic with  $I$  and  $J$  as triple points. Thus we see that our transformation corresponds to one in the projective plane which sends a line into an octavic with seven triple points. This is the well-known Geiser involution, and by analogy we call Type C the Geiser involution in the inversive plane. We can now restate our main theorem thus:

*There is a Geiser involution for which the six biquadratics are all rectangular. This requires three conditions on the six points.*

*Note on Curves in the Inversive Plane.*

For the sake of those readers who are not familiar with the handling of curves in the inversive plane, I append the following note:

In this plane we use as co-ordinates  $z = x + iy$  and  $\bar{z} = x - iy$  ( $x$  and  $y$  being the ordinary Cartesian co-ordinates of a point.) These are variously called minimal, circular or absolute co-ordinates. A curve is given by an expression in  $z$  and  $\bar{z}$ , equated to zero. It is generally written in the form of a self-conjugate matrix such as

$$\begin{array}{c|cccccccc}
 & 1 & z & z^2 & z^3 & z^4 & \dots & z^n \\
 \hline
 1 & P_0 & a_1 & a_2 & a_3 & a_4 & \dots & a_n \\
 \bar{z} & \bar{a}_1 & P_1 & b_1 & b_2 & b_3 & \dots & b_{n-1} \\
 \bar{z}^2 & \bar{a}_2 & \bar{b}_1 & P_2 & c_1 & c_2 & \dots & c_{n-2} \\
 \bar{z}^3 & \bar{a}_3 & \bar{b}_2 & \bar{c}_1 & P_3 & d_1 & \dots & d_{n-3} \\
 \bar{z}^4 & \bar{a}_4 & \bar{b}_3 & \bar{c}_2 & \bar{d}_1 & P_4 & & \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\
 \bar{z}^n & \bar{a}_n & \bar{b}_{n-1} & \bar{c}_{n-2} & \bar{d}_{n-3} & & & P_n
 \end{array} = 0,$$

where the  $p$ 's are real and the other numbers are complex quantities and their conjugates. This is read by multiplying every element by the quantities heading the row and column in which the element stands and adding all such products together. Thus, the curve written above would start off as

$$P_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots + a_nz^n \\ + \bar{a}_1\bar{z} + P_1z\bar{z} + b_1z^2\bar{z} + b_2z^3\bar{z} + \cdots = 0.$$

A curve is always named after the highest powers of  $z$  and  $\bar{z}$  occurring in it. For instance, if the term of highest power is  $p_nz^n\bar{z}^n$  the curve is called a  $2n$ -ic, and is denoted by  $C_n$ . It is easily seen that this curve is, projectively, a  $2n$ -ic, with an  $n$ -fold point at  $I$  and an  $n$ -fold point at  $J$ , since if we express  $p_nz^n\bar{z}^n$  in Cartesian co-ordinates it becomes  $p_n(x^2 + y^2)^n$  and this is the leading term of the curve when written in the ordinary way. The simplest curve is the  $C_1$

$$\begin{array}{c} 1 \quad z \\ 1 \left| \begin{array}{cc} P_0 & a_1 \\ \bar{z} & \bar{a}_1 \end{array} \right. P_1 \end{array} = 0,$$

or  $p_0 + a_1z + \bar{a}_1\bar{z} + p_1z\bar{z} = 0$ , which is a circle, since it is a conic on  $I$  and  $J$ . The next simplest is the  $C_2$  or biquadratic

$$\begin{array}{c} 1 \quad z \quad z^2 \\ 1 \left| \begin{array}{ccc} P_0 & a_1 & a_2 \\ \bar{z} & \bar{a}_1 & P_1 \\ \bar{z}^2 & \bar{a}_2 & \bar{b}_1 \end{array} \right. P_2 \end{array} = 0,$$

or

$$P_0 + a_1z + \bar{a}_1\bar{z} + a_2z^2 + \bar{a}_2\bar{z}^2 + P_1z\bar{z} + b_1z^2\bar{z} + \bar{b}_1\bar{z}^2z + P_2z^2\bar{z}^2 = 0$$

which is projectively a quartic with  $I$  and  $J$  as double points.

If, in a  $C_n$ , the coefficient of  $z^n\bar{z}^n$  is zero, the  $C_n$  passes through infinity. If, in addition, the coefficients of  $z^n\bar{z}^{n-1}$  and  $\bar{z}^n z^{n-1}$  are both zero, the curve has a double point at infinity. Thus, writing the curve in matrix form, we see that by striking off the element in the lower right-hand corner, we make the curve pass through infinity. This causes the curve to be, projectively speaking, a  $(2n-1)$ ic, with  $I$  and  $J$  as  $(n-1)$  fold points, for our highest term will be of the form

$$kz^n\bar{z}^{n-1} + \bar{k}\bar{z}^nz^{n-1} = z^{n-1}\bar{z}^{n-1}[z(k_1 + ik_2) + \bar{z}(k_1 - ik_2)] \\ = z^{n-1}\bar{z}^{n-1}[(k_1 + ik_2)(x + iy) + (k_1 - ik_2)(x - iy)] \\ = (x^2 + y^2)^{n-1}[2k_1x - 2k_2y].$$

Taking off in addition the next two elements in the lower right-hand corner makes infinity a double point, and makes the curve equivalent to a curve in the projective plane of order  $2n - 2$ , with  $(n - 2)$ -fold points at  $I$  and  $J$ . We see immediately that a circle on infinity is a straight line, while a  $C_2$  on infinity is a circular cubic. A  $C_2$  with a double point at infinity is a conic.

In general, we may say that starting in the lower right-hand corner of the matrix and striking off  $p$  diagonals in succession makes our  $C_n$  have a  $p$ -fold point at infinity; it is a curve in the projective plane of order  $(2n - p)$  with  $(n - p)$  fold points at  $I$  and  $J$ . Given a curve in the inversive plane whose highest terms are of the form  $kz^p\bar{z}^q$  and  $\bar{k}\bar{z}^p z^q$ , where  $p \neq q$ , then the curve is a  $C_n$ , where  $n$  is the greater of the two integers  $p$  and  $q$ , which passes through infinity  $p - q$  or  $q - p$  times. It is, in the projective plane a curve of order  $p + q$  which passes through  $I$  and  $J$  each either  $q$  or  $p$  times. Thus, a curve whose highest terms are of the form  $z^2\bar{z}$  and  $\bar{z}^2z$  is a  $C_2$  with infinity as a simple point. It is projectively a curve of the third degree which passes simply through  $I$  and  $J$ ; i. e., a circular cubic.

A  $C_n$  with an  $n$ -fold point at infinity is a curve in the projective plane of order  $n$ , not passing through  $I$  and  $J$ . Accordingly we may say that an ordinary curve of order  $n$  in the projective plane is, in the inversive plane, a  $C_n$  with an  $n$ -fold point at infinity. A curve of order  $n$  in the projective plane which passes through  $I$  and  $J$  each  $m$  times becomes, in the inversive plane, a  $C_{n-m}$  with infinity as  $(n - 2m)$ -fold point, whose highest terms are of the form  $z^{n-m}\bar{z}^m$  and  $z^m\bar{z}^{n-m}$ . Thus, a curve of order nine in the projective plane, which passes through  $I$  and  $J$  each three times, becomes a  $C_6$  with infinity as a triple point, whose highest terms are of the form  $z^6\bar{z}^3$  and  $\bar{z}^6z^3$ .

One more point concerning these curves must be called to the attention of the reader: A  $C_n$  and a  $C_m$  have in common  $2mn$  points.\* For the  $C_n$ , being a  $2n$ -ic with  $n$ -fold points at  $I$  and  $J$ , and the  $C_m$  being a  $2m$ -ic with  $m$ -fold points at  $I$  and  $J$ , intersect in  $4mn$  points; but the intersections at  $I$  and  $J$  use up  $2mn$  of these, so that, as a consequence, we may say that they have in common only  $4mn - 2mn = 2mn$  points.

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\* The common points are either actual intersections or common image-pairs.



## VITA.

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Mildred Caroline Waters was born in Baltimore, Maryland, on February 17th, 1904. She received her early education in public and private schools of that city, and at Goucher College, where she received the A. B. degree in 1924. During the year 1924-1925 she was a graduate student in chemistry at the Johns Hopkins University, and during the years 1925-1926 and 1926-1927 a graduate student in mathematics at the same university. In the mathematics department she studied under Drs. Morley, Cohen, Murnaghan, Musselman, Rainich and Nelson. She received the degree of M. A. in mathematics in June, 1927. In September 1927 she was married to C. E. Dean and moved to New York City. Here she studied mathematics at Columbia University, during the academic year 1927-1928, and also from October to December 1928. While studying at Columbia, she still remained in contact with the Johns Hopkins University, and continued work on a dissertation which she had begun there under the direction of Professor Morley. This dissertation was completed in October, 1928. She received the Ph. D. degree from the Johns Hopkins University in February, 1929.







